

Parity-relevant Zitterbewegung and quantum simulation by a single trapped ion

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Zitterbewegung (ZB), the trembling of free relativistic electrons in a vacuum could be simulated by a single trapped ion. We focus on the variations of ZB under different parity conditions and find no ZB in the case of odd or even parity. ZB occurs only for admixture of the odd and even parity states. We also show the similar role played by the parity operator for the trapped ion in Fock-state representation and the space inversion operator for a realistic relativistic electron. Although the ZB effect is invisible in a relativistic electron, preparation of the trapped ion in different parity states is a sophisticated job, which makes it possible to observe the parity relevant ZB effects with currently available techniques.

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Since the discovery by Schrödinger [1], Zitterbewegung (ZB), i.e., the trembling of free relativistic electrons, has drawn more and more attention and interests over past years [2–8]. It has been generally believed that the trembling of a relativistic electron is resulted by the interference between negative and positive energy components, a typical relativistic feature of the Dirac electron. Up to now, however, no direct observation of ZB has been achieved due to inaccessibility with current experimental techniques, which led to some questioning on the ZB [6, 9]. Moreover, there have been alternative explanations for the origin of the ZB, such as the continuously virtual transition process between different internal states in view of quantum field theory [8] or the relevance to the complex phase factor in context of space-time algebra [3].

By quantum simulation, some relativistic effects have recently been demonstrated in some controllable physical systems, such as graphene, semiconductor, superconductor and trapped ion [10–18]. It was recently considered that the ZB occurs not only under the relativistic condition, but also extensively in the dynamics of a system with more than one degree of freedom [14, 19]. The demonstration of the ZB beyond the relativistic electron helps us to further understand the Dirac equation and the relevant relativistic phenomena.

In this work, we focus on the role of parity played in the simulation of Dirac equation for the ZB effect by a single trapped ion. Like in [10, 16], we also employ the motional degrees of freedom of the ion to simulate the position and momentum of the relativistic electron, and the internal degrees of freedom to refer to the energy states. Since the motional state of the ion could be quantized, we may discuss the problem in number-state representation. The key point of our work is to introduce a parity operator $\hat{\Pi}$, by which we show that, besides the conventional consid-

eration of the origin of the ZB, i.e., the interference between the positive- and negative-energy components, the ZB is also relevant to parity of the states. To understand the relevant physics, we will compare the parity operator of the trapped ion with the space inversion operator of the realistic relativistic electron. The experimental feasibility to observe the parity-relevant ZB effects will be justified.

Under the radiation of three laser lights with red-detuning, blue-detuning, and carrier transition, respectively, the interacting Hamiltonian of a single trapped ion reads [10, 16]

$$H = i\hbar\eta\tilde{\Omega}(a^+ - a)\hat{\sigma}_x + \hbar\Omega\sigma_z, \quad (1)$$

where η is the Lamb-Dicke parameter, $\tilde{\Omega}$ and Ω are, respectively, the effective Rabi frequency and the effective Larmor frequency of the ion. a^+ (a) is the creation (annihilation) operator of the quantized motion of the ion. σ_x and σ_z are usual Pauli operators. Defining $p = i\hbar(a^+ - a)/2\Delta$ with Δ the size of the ground state wave function, we may rewrite the Hamiltonian as a form analogous to a 1 + 1 dimensional Dirac equation

$$H_D = 2\eta\Delta\tilde{\Omega}p\sigma_x + \hbar\Omega\sigma_z \quad (2)$$

provided $c := 2\eta\Delta\tilde{\Omega}$ and $mc^2 := \hbar\Omega$.

Introducing the parity operator

$$\hat{\Pi} = e^{i\pi(a^+a - \frac{1}{2} + \frac{1}{2}\sigma_z)} \quad (3)$$

commuting with the Hamiltonian in Eq. (1), which means $\hat{\Pi}$ is a conserved quantity under the Hamiltonian \hat{H} , we study below the dynamics of the trapped ion with different parity states, i.e., of the odd or even parity, or of admixture of the both.

The ZB is related to the average position $\langle x(t) \rangle$ of the trapped ion. Since our interest is in the states under some parity conditions, we have to find the common eigenfunctions of \hat{H} and $\hat{\Pi}$. To this end, we define $|p\rangle$ as the eigenfunction of $(a^+ - a)$ by assuming $(a^+ - a)|p\rangle = -ip|p\rangle$.

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Straightforward deduction yields

$$|p\rangle = \frac{1}{\sqrt{2\pi}} e^{-\frac{p^2}{4}} \sum_{n=0}^{\infty} \frac{i^n H_n(p)}{\sqrt{n!}} |n\rangle, \quad (4)$$

where $H_n(x)$ is a Hermite polynomial defined by $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$ [20]. It could be proven that the states in $\{|p\rangle\}$ are orthonormal and complete, and each state $|p\rangle$ corresponds to the momentum $p_{mom} = \hbar p/(2\Delta)$.

Substituting Eq.(4) into Eq.(1) and diagonalizing \hat{H} yields the eigenvalues

$$E_{\pm} = \pm \sqrt{(\hbar\Omega)^2 + (\eta\hbar\tilde{\Omega}p)^2}$$

and the eigenstates

$$\begin{aligned} |S_{E_+}(p)\rangle &= N(p) \left[1, \frac{\eta\hbar\tilde{\Omega}p}{|E_+| + \hbar\Omega} \right]^T, \\ |S_{E_-}(p)\rangle &= N(p) \left[\frac{-\eta\hbar\tilde{\Omega}p}{|E_-| + \hbar\Omega}, 1 \right]^T, \end{aligned}$$

with $N(p) = \sqrt{(|E_{\pm}| + \hbar\Omega)/2|E_{\pm}|}$ the normalization factor. So the total eigenfunctions of \hat{H} are $|\psi_{E_+}(p)\rangle = |S_{E_+}(p)\rangle \otimes |p\rangle$ and $|\psi_{E_-}(p)\rangle = |S_{E_-}(p)\rangle \otimes |p\rangle$.

Under the parity operator $\hat{\Pi}$, we assume following parity states in the number-state representation,

$$|o\rangle = |+\rangle \sum_m f_{2m+1} |2m+1\rangle + |-\rangle \sum_m f_{2m} |2m\rangle, \quad (5)$$

$$|e\rangle = |+\rangle \sum_m f_{2m} |2m\rangle + |-\rangle \sum_m f_{2m+1} |2m+1\rangle, \quad (6)$$

where f_k ($k = 2m, 2m+1, m = 0, 1, 2, \dots$) are normalized coefficients. $|o\rangle$ stands for odd parity state due to $\hat{\Pi}|o\rangle = -|o\rangle$ and $|e\rangle$ for even parity state with $\hat{\Pi}|e\rangle = |e\rangle$. $|\pm\rangle$ are the eigenstates of σ_z with eigenvalues ± 1 , respectively.

For our purpose, we may construct the spin states $|\pm\rangle$ by positive energy eigenstates as

$$|+\rangle = W(|S_{E_+}(p)\rangle + |S_{E_+}(-p)\rangle), \quad (7)$$

$$|-\rangle = W'(|S_{E_+}(p)\rangle - |S_{E_+}(-p)\rangle), \quad (8)$$

or by negative energy eigenstates as,

$$|+\rangle = W'(|S_{E_-}(-p)\rangle - |S_{E_-}(p)\rangle), \quad (9)$$

$$|-\rangle = W(|S_{E_-}(p)\rangle + |S_{E_-}(-p)\rangle), \quad (10)$$

with normalization factors $W = \sqrt{|E_{\pm}|/(2|E_{\pm}| + 2\hbar\Omega)}$ and $W' = \sqrt{[|E_{\pm}|(|E_{\pm}| + \hbar\Omega)]/2(\eta\hbar\tilde{\Omega}P)^2}$. Moreover, the odd and even motional states are associated with the momentum eigenstates $|p\rangle$, i.e.,

$$\sum_m f_{2m+1} |2m+1\rangle \propto \frac{1}{\sqrt{2}}(|p\rangle - |-p\rangle), \quad (11)$$

$$\sum_m f_{2m} |2m\rangle \propto \frac{1}{\sqrt{2}}(|p\rangle + |-p\rangle). \quad (12)$$

Substituting Eqs. (7), (8), (11) and (12) into Eq. (5), we may write down the co-eigenstate of E_+ and odd parity to be,

$$\begin{aligned} |\psi_{E_+}^o\rangle &= W(p)(|S_{E_+}(p)\rangle + |S_{E_+}(-p)\rangle) \otimes \frac{1}{\sqrt{2}}(|p\rangle - |-p\rangle) \\ &+ QW'(p)(|S_{E_+}(p)\rangle - |S_{E_+}(-p)\rangle) \otimes \left(\frac{1}{\sqrt{2}}(|p\rangle + |-p\rangle)\right) \end{aligned}$$

where Q is a coefficient to be determined. We only keep some reasonable terms by setting $Q = W(p)/W'(p)$, i.e., elimination of the terms of $|S_{E_+}(p)\rangle |-p\rangle$ and $|S_{E_+}(-p)\rangle |p\rangle$. Then we have,

$$|\psi_{E_+}^o\rangle = \frac{1}{\sqrt{2}} [|\psi_{E_+}(p)\rangle - |\psi_{E_+}(-p)\rangle]. \quad (13)$$

Similarly, we have other co-eigenstates of E_+ and even parity, E_- and different parities [21],

$$|\psi_{E_+}^e\rangle = \frac{1}{\sqrt{2}} [|\psi_{E_+}(p)\rangle + |\psi_{E_+}(-p)\rangle], \quad (14)$$

$$|\psi_{E_-}^o\rangle = \frac{1}{\sqrt{2}} [|\psi_{E_-}(p)\rangle + |\psi_{E_-}(-p)\rangle], \quad (15)$$

$$|\psi_{E_-}^e\rangle = \frac{1}{\sqrt{2}} [|\psi_{E_-}(p)\rangle - |\psi_{E_-}(-p)\rangle]. \quad (16)$$

Based on these eigenstates, the average position of the ion $\langle x(t) \rangle = \langle (\hat{a}^+ + a)\Delta \rangle$ can be calculated by the evolved states with odd or even parity, where

$$\begin{aligned} |\psi^o(t)\rangle &= \sum_p a_p |\psi_{E_+}^o(p)\rangle e^{-\frac{i|E_+|t}{\hbar}} + \sum_p b_p |\psi_{E_-}^o(p)\rangle e^{\frac{i|E_-|t}{\hbar}} \\ |\psi^e(t)\rangle &= \sum_p a_p |\psi_{E_+}^e(p)\rangle e^{-\frac{i|E_+|t}{\hbar}} + \sum_p b_p |\psi_{E_-}^e(p)\rangle e^{\frac{i|E_-|t}{\hbar}}, \end{aligned}$$

with a_p and b_p the coefficients determined by the initial condition.

To calculate $\langle x(t) \rangle$, we first consider $\langle dx/dt \rangle$, the average velocity,

$$\left\langle \frac{dx}{dt} \right\rangle = \frac{\Delta}{i\hbar} \left\langle [\hat{a}^+ + \hat{a}, \hat{H}_D] \right\rangle = 2\eta\Delta\tilde{\Omega} \langle \hat{\sigma}_x \rangle.$$

Taking the odd parity as an example, we have

$$\begin{aligned} \langle \psi^o(t) | \hat{\sigma}_x | \psi^o(t) \rangle &= \sum_p a_p^* a_p \langle \psi_{E_+}^o(t) | \hat{\sigma}_x | \psi_{E_+}^o(t) \rangle \\ &+ \sum_p b_p^* b_p \langle \psi_{E_-}^o(t) | \hat{\sigma}_x | \psi_{E_-}^o(t) \rangle \\ &+ \sum_p a_p^* b_p e^{\frac{2i|E_{\pm}|t}{\hbar}} \langle \psi_{E_+}^o(t) | \hat{\sigma}_x | \psi_{E_-}^o(t) \rangle \\ &+ \sum_p b_p^* a_p e^{\frac{-2i|E_{\pm}|t}{\hbar}} \langle \psi_{E_-}^o(t) | \hat{\sigma}_x | \psi_{E_+}^o(t) \rangle. \end{aligned} \quad (17)$$

It is easy to check in Eq. (17) that $\langle \psi^o(t) | \hat{\sigma}_x | \psi^o(t) \rangle = 0$ since every term is zero. As a result, $\langle x(t) \rangle$ remains unchanged. Similarly we have $\langle \psi^e(t) | \hat{\sigma}_x | \psi^e(t) \rangle = 0$. These results imply that a trapped ion in an eigenstate of $\hat{\Pi}$ would be on average static.

Because Eq. (17) involves both the positive and negative energy components, the ZB should occur, according to the conventional viewpoint, due to their interference. Our result, however, presents that the ZB depends not only on the interference between the positive and negative energy components, but also on parity of the states.

To be more clarified, we have numerically calculated in Fig. 1 the average position of the trapped ion with the initial state $(\cos \beta |+\rangle + \sin \beta |-\rangle) |0\rangle$. Since both $|+\rangle |n\rangle$ and $|-\rangle |n\rangle$ are states with definite parity, the case with $\alpha = \beta$ is of the most mixed parity. By changing the values of β , we may see clearly from the figure that the ZB occurs only in the admixture of the odd and even parity eigenstates.

Our result could be understood by the viewpoint in [19] that the ZB could appear in any system, besides the relativistic system, with more than one degree of freedom. In our case, the ZB effect is originated from the interplay between the internal and motional degrees of freedom of the ion. With respect to the conventional viewpoint of interference between the positive and negative energy components, we may check Eq. (17) again which involves both positive and negative energy states. The internal-motional-state interplay leads to the interference between different energy component terms. Once the ion is in a certain parity state, however, the interference is destructive, yielding $\langle x \rangle = 0$. So the ZB appears only in the case of admixture of different parity states, which allows the constructive interference between different energy components. Simply speaking, whether the ZB appears or not, is decided by both the interference and symmetry, the latter of which is reflected by parity.

For a realistic relativistic electron, there is no demand to quantize the motional freedom, but the parity operator $\hat{\Pi}$ discussed above reminds us of the space inversion operator \hat{P} defined as [22],

$$\hat{P}\varphi(\bar{x}, t) = \hat{\sigma}_z \varphi(-\bar{x}, t), \quad (18)$$

where $\varphi(\bar{x}, t)$ is the wavefunction of the relativistic electron, and we have used the overline to represent the parameters of the relative electron in order to distinguish from the ones of the trapped ion. For a relativistic electron, the eigenstates of \hat{H}_D are,

$$\begin{aligned} \varphi_{\bar{E}_+}(\bar{p}; \bar{x}, t) &= N(\bar{p}) \left(\frac{1}{|E_+| + \bar{m}c^2} \right) e^{\frac{i\bar{p}\bar{x}}{\hbar} - \frac{i|E_+|t}{\hbar}} \\ \varphi_{\bar{E}_-}(\bar{p}; \bar{x}, t) &= N(\bar{p}) \left(\frac{-c\bar{p}}{|E_-| + \bar{m}c^2} \right) e^{\frac{i\bar{p}\bar{x}}{\hbar} + \frac{i|E_-|t}{\hbar}}, \end{aligned}$$

where \bar{m} is the mass of the electron, \bar{p} and \bar{x} are, respectively, the momentum and the position, \bar{E}_\pm stand for the energies and $N(\bar{p})$ the normalization factor.

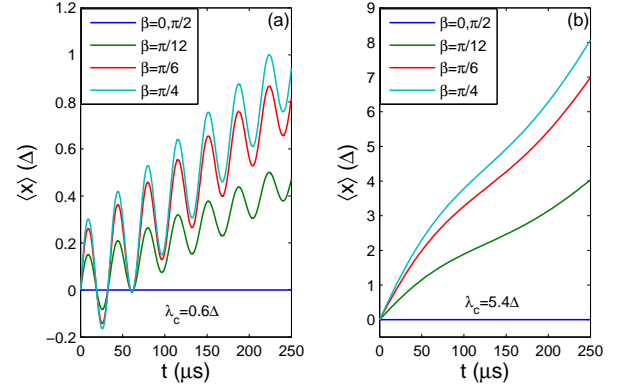


FIG. 1: (Color online) Numerical simulation of $\langle \hat{x}(t) \rangle$ in units of Δ for initial state $(\cos \beta |+\rangle + \sin \beta |-\rangle) |0\rangle$, where $|\pm\rangle$ are eigenstates of σ_z and $|0\rangle$ is the ground motional state. $\lambda_c = 2\eta\Omega\Delta/\Omega$ is the Compton wavelength. We consider (a) $\lambda_c = 0.6\Delta$, near the non-relativistic limit and (b) $\lambda_c = 5.4\Delta$ close to the relativistic limit. From the bottom to top, the curves correspond to $\beta = 0$ or $\pi/2$, $\pi/12$, $\pi/6$ and $\pi/4$, respectively.

We have $\hat{P}\varphi_{\bar{E}_\pm}(\bar{p}; \bar{x}, t) = \pm\varphi_{\bar{E}_\pm}(-\bar{p}; \bar{x}, t)$. Under such a consideration, we may easily find the odd parity states

$$\begin{aligned} \varphi_{\bar{E}_+}^o &= \frac{1}{\sqrt{2}}[\varphi_{\bar{E}_+}(\bar{p}) - \varphi_{\bar{E}_+}(-\bar{p})], \\ \varphi_{\bar{E}_-}^o &= \frac{1}{\sqrt{2}}[\varphi_{\bar{E}_-}(\bar{p}) + \varphi_{\bar{E}_-}(-\bar{p})], \end{aligned}$$

and the even parity states

$$\begin{aligned} \varphi_{\bar{E}_+}^e &= \frac{1}{\sqrt{2}}[\varphi_{\bar{E}_+}(\bar{p}) + \varphi_{\bar{E}_+}(-\bar{p})], \\ \varphi_{\bar{E}_-}^e &= \frac{1}{\sqrt{2}}[\varphi_{\bar{E}_-}(\bar{p}) - \varphi_{\bar{E}_-}(-\bar{p})]. \end{aligned}$$

They are formally consistent with the co-eigenstates Eqs. (13)-(16) for the trapped ion. In this sense, we consider that $\hat{\Pi}$ and \hat{P} play the same role in the respective systems. In other words, $\hat{\Pi}$ is something like an inversion parity operator in the number state representation, which moves population back and forth between the internal and motional states.

Following the same steps as for the trapped ion, we can immediately find $\langle \bar{x} \rangle = 0$ of the relativistic electron under definite parity states. So the ZB of the relativistic electron occurs when the electron is in both the admixture of different parity states and the admixture of different energy components. A particle, such as a relativistic electron or a trapped ion, staying in a certain parity state will be static or in harmonic oscillation. Otherwise, a moving particle with different energy components coexisting will surely experience the ZB.

Since the ZB in a real relativistic electron is inaccessible with current technique, we have to resort to other systems, such as the trapped ion, to observe the parity-relevant ZB effect. With different trapped ion's motional

states, the eigenstates of the parity operator have abundant forms, such as $|\pm\rangle \otimes |n\rangle$ and $|+\rangle \otimes |n\rangle_A \pm |-\rangle \otimes |n\rangle_B$ where $|n\rangle_{A(B)}$ is the displaced coherent state [23]. With currently available ion-trap technique, it has already been achieved the cooling of the single ion to the motional ground state [24]. Preparation of the ion to a certain motional Fock state or a certain coherent state has also been a sophisticated job [25]. As a result, using the parity states mentioned above, we may check the variation of the ZB with respect to different parity conditions following the operations in [16].

Besides the simulation of Dirac equation, the parity operator $\hat{\Pi}$ also has application in quantum computing. $\hat{\Pi}$ commutes not only with \hat{H} , but also with other laser-ion interaction Hamiltonians, such as $\hat{H}_r = \sigma_+ a + \sigma_- a^\dagger$, a usually used Hamiltonian for logic gate operation [26]. If we prepare the ion in the state $|+\rangle|0\rangle$ or other parity states, the ion will be more steady during the operation

than in any mixed parity states. This is advantageous to quantum gate operation.

In summary, we have investigated the dynamics of a single trapped ion under some certain parity conditions. Our study has shown that the ZB in the trapped ion is relevant not only to the interference between different energy components, but also to parity. To understand the physics related to the realistic Dirac particle, we have discussed the correspondence of the parity between a relativistic electron and a trapped ion. Experimental feasibility of observing our results by a trapped ion has been justified with currently available techniques. We argue that our study is not only helpful to explore the ZB effect itself, but also useful to further understand the quantum characteristic of the ultracold trapped ion.

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